ON THE REPRESENTATION OF A NUMBER AS THE SUM OF ANY NUMBER OF SQUARES, AND IN PARTICULAR OF FIVE*

BY

G. H. HARDY

1. Introduction

1. 1. In a short note published recently in the Proceedings of the National Academy of Sciences[1] I sketched the outlines of a new solution of one of the most interesting and difficult problems in the Theory of Numbers, that of determining the number of representations of a given integer as the sum of five or seven squares. The method which I use is one of great power and generality, and has been applied by Mr. J. E. Littlewood, Mr. S. Ramanujan, and myself to the solution of a number of different problems; and it is probable that, in our previous writings on the subject,[2] we have explained sufficiently the general ideas on which it rests. I may therefore confine myself, for the most part, to filling in the details of my previous work. I should observe, however, that the method by which I now sum the "singular series", which plays a dominant rôle in the analysis,

* Presented to the Society, February, 1920.

[1] G. H. Hardy, *On the expression of a number as the sum of any number of squares, and in particular of five or seven*, Proceedings of the National Academy of Sciences, vol. 4 (1918), pp. 189–193.

[2] G. H. Hardy and S. Ramanujan: (1) *Une formule asymptotique pour le nombre des partitions de n*, Comptes Rendus, 2 Jan. 1917; (2) *Asymptotic formulae in Combinatory Analysis*, Proceedings of the London Mathematical Society, ser. 2, vol. 17 (1918), pp. 75–115; (3) *On the coefficients in the expansions of certain modular functions*, Proceedings of the Royal Society, (A), vol. 95 (1918), pp. 144–155:

S. Ramanujan, *On certain trigonometrical sums and their applications in the theory of numbers*, Transactions of the Cambridge Philosophical Society, vol. 22 (1918), pp. 259–276:

G. H. Hardy and J. E. Littlewood: (1) *A new solution of Waring's Problem*, Quarterly Journal of Mathematics, vol. 48 (1919), pp. 272–293: (2) *Note on Messrs. Shah and Wilson's paper entitled On an empirical formula connected with Goldbach's Theorem*, Proceedings of the Cambridge Philosophical Society, vol. 19 (1919), pp. 245–254: (3) *Some problems of 'Partitio Numerorum', (I) A new solution of Waring's Problem*, Göttinger Nachrichten, 1920: (4) *Some problems of 'Partitio Numerorum', (II) Proof that every large number is the sum of at most 21 biquadrates*, Mathematische Zeitschrift, 1920.

The two last papers will be published shortly.

is quite different from that which I sketched in my former note. The new method has important applications to a whole series of problems in Combinatory Analysis, concerning the representation of numbers by sums of squares, cubes, kth powers, or primes. It is in the present problem that it finds its simplest and most elegant application, and it is most instructive to work this application out in detail.

It is well known that the solution of the problem is a good deal simpler when s, the number of squares in question, does not exceed 8. If s is 2, 4, 6, or 8, the number of representations may be expressed in finite form by means of the real divisors of n; if s is 3, 5, or 7, by means of quadratic residues and non-residues. If $s > 8$, other and more recondite arithmetical functions are involved. In this paper I confine myself to the cases in which $s \leqq 8$. Among these, those in which s is odd have always been regarded as notably the more difficult, and one of my principal objects has been to place them all upon the same footing. But I generally suppose $s = 5$ or $s = 8$, cases typical of the odd and even cases respectively.

In Section 2 I construct the *singular series*

$$\frac{\pi^{\frac{1}{2}s}\, n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)} \sum_{1}^{\infty} A_k,$$

where

$$A_1 = 1, \qquad A_k = \frac{1}{k^s} \sum_{h} (S_{h,k})^s\, e^{-2nh\pi i/k};$$

$S_{h,k}$ denoting the Gaussian sum[3]

$$\sum_{j=1}^{k} e^{2j^2 h\pi i/k},$$

and the summation extending over all positive values of h less than and prime to k. The series may be written in the form

$$\frac{\pi^{\frac{1}{2}s}\, n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)} \Big[1 + 0 + \frac{2}{3^{\frac{1}{2}s}} \cos\left(\tfrac{2}{3}n\pi - \tfrac{1}{2}s\pi\right) + \frac{2^{\frac{1}{2}s+1}}{4^{\frac{1}{2}s}} \cos\left(\tfrac{1}{2}n\pi - \tfrac{1}{4}s\pi\right)$$
$$+ \frac{2}{5^{\frac{1}{2}s}} \{\cos \tfrac{2}{5} n\pi + \cos\left(\tfrac{4}{5}n\pi - s\pi\right)\} + 0 + \cdots \Big],$$

the zero terms corresponding to $k = 2$ and $k = 6$.

In Section 3 I show that, when $s = 8$ or $s = 5$, the sum of the singular series is in fact $r_s(n)$, the number of representations of n as a sum of s squares. The methods used are equally applicable in the cases of 3, 4, 6, or 7 squares;

[3] In my former note I denoted a typical "rational point" on the unit circle by $e^{h\pi i/k}$, and a typical Gaussian sum by

$$\sum_{0}^{k-1} e^{j^2 h\pi i/k}.$$

In this paper I generally use the forms involving a 2. Each notation has special advantages for particular purposes.

but the case of two squares is abnormal.[4] Throughout this section I am very deeply indebted to a paper by Mr. Mordell, published recently in the Quarterly Journal of Mathematics.[5] My proof of the identity of the functions which I call ϑ^s and Θ_s is in fact based directly on his work. It is true that Mordell considers only the case in which s is even; but his argument is applicable in principle to either case, and was applied by him to the even case only merely because, at the time when his paper was written, he had no method for the construction, when s is odd, of the essential "principal invariant" denoted by him by χ. It is the construction of this invariant by a uniform method in all cases, through the medium of the "singular series", that is my own principal contribution to the subject.

In Section 4 I show how the singular series may be transformed into a product, and give general rules for the calculation of the terms of the product. All the results of this section are independent of the hypothesis $s \leqq 8$. In Section 5 I sum the series when $s = 8$, and obtain Jacobi's well-known results. In Section 6 I consider the case $s = 5$, supposing however that n has no squared factor, so that there is no distinction between primitive and imprimitive representations; and I obtain results equivalent to those enunciated first by Eisenstein and proved later by Smith and Minkowski. In Section 7 I consider the general case, and show that the method leads to the more complete results of Smith. I conclude, in Section 8, by some remarks as to the application of the method when $s > 8$. I do not pursue this subject further because such applications belong more naturally, either to Mr. Littlewood's and my own researches in connection with Waring's problem, or to Mr. Mordell's in connection with the general theory of modular invariants.

It will be noticed that the explicit formulas for the powers of the fundamental theta-function, such as the familiar formula

$$\vartheta^8 = (1 + 2q + 2q^4 + \cdots)^8 = 1 + 16\left(\frac{1^3 q}{1+q} + \frac{2^3 q^2}{1-q^2} + \frac{3^3 q^3}{1+q^3} + \cdots\right),$$

or the new formula[6]

$$\vartheta^5 = 1 + \frac{32}{3}\left\{ \sum_{1,3,5,\ldots} \frac{1}{k^2} \sum_j \sum_{m=0}^{\infty} (mk+j)^{3/2} q^{mk+j} - \sum_{2,4,6,\ldots} \frac{1}{k^2} \sum_j \sum_{m=0}^{\infty} (-1)^{m+\mu} (mk+j)^{3/2} q^{mk+j} \right\},$$

do not appear at all in my present analysis.

[4] See Mr. Ramanujan's paper quoted in footnote 2.

[5] L. J. Mordell, *On the representations of numbers as a sum of 2r squares*, Quarterly Journal of Mathematics, vol. 48 (1917), pp. 93–104. See also a later paper by the same author, *On the representations of a number as a sum of an odd number of squares*, Transactions of the Cambridge Philosophical Society, vol. 22 (1919), no. 17, pp. 361–372.

[6] This is formula (10) of my former note, where the meaning of j and μ is explained. See also p. 360 of Mr. Mordell's second paper cited above.

In the sequel I give references only to isolated results directly required for the objects of my analysis. It is more convenient to collect here some notes concerning the older memoirs dealing with the problem.

Jacobi's classical results concerning 2, 4, 6, or 8 squares are quoted by Smith on p. 307 of his *Report on the Theory of Numbers* (*Collected Papers*, vol. 1). They are contained implicitly in §§ 40–42 of the *Fundamenta Nova* (pp. 103–115).

Liouville gave formulas relating to the cases of 10 and 12 squares in a number of short notes in the second series of the Journal des mathématiques: see in particular vol. 5, p. 143; vol. 6, p. 233; vol. 9, p. 296; vol. 10, p. 1. These notes appeared between 1860 and 1865.

Later Glaisher, in a series of papers published in the Quarterly Journal, worked out systematically all cases in which s is even and between 2 and 18 inclusive. He has given a short summary of his results in a paper *On the numbers of representations of a number as a sum of* $2r$ *squares, where* $2r$ *does not exceed* 18, published in the Proceedings of the London Mathematical Society, ser. 2, vol. 5 (1907), pp. 479–490. This paper contains full references to his more detailed work.

The results for 5 squares (for numbers which have no square divisors) were stated without proof by Eisenstein on p. 368 of vol. 35 (1847) of Crelle's Journal. They were completed by Smith, who stated the general results at the end of his memoir *On the orders and genera of quadratic forms containing more than three indeterminates* (Proceedings of the Royal Society, vol. 13 (1864), pp. 199–203, and vol. 16 (1867), pp. 197–208; *Collected Papers*, vol. 1, pp. 412–417, 510–523). No detailed proofs, however, appeared before the publication of the prize memoirs of Smith (*Mémoire sur la représentation des nombres par des sommes de cinq carrés*, Mémoires présentés par divers savants à l'Académie, vol. 29, no. 1 (1887), pp. 1–72; *Collected Papers*, vol. 2, pp. 623–680) and Minkowski (*Mémoire sur la théorie des formes quadratiques à coefficients entières*, ibid., no. 2, pp. 1–178; *Gesammelte mathematische Abhandlungen*, vol. 1, pp. 3–144).

The methods for the summation of the series

$$\sum \left(\frac{n}{m}\right)\frac{1}{m^2},$$

which is fundamental in the five square problem, and other series of similar type, are due to Dirichlet (*Recherches sur divers applications de l'analyse infinitésimale à la théorie des nombres*, Crelle's Journal, vol. 19 (1839), pp. 324–369, and vol. 21 (1840), pp. 1–12, 134–155; *Werke*, vol. 1, pp. 411–497) and to Cauchy (*Mémoire sur la théorie des nombres*, Mémoires de l'Académie des Sciences, vol. 17 (1840), pp. 249–768; especially Note 12, pp. 665–699).

A systematic account of the whole theory is given by Bachmann in vol. 4 of his *Zahlentheorie.* Bachmann works out the case $s = 7$ also in detail.

2. Formal construction of the singular series

2. 1. I write, as in my former note

$$(2.11)\quad f(q) = 1 + \sum_{1}^{\infty} r_s(n) q^n = (1 + 2q + 2q^4 + \cdots)^s = \{\vartheta_3(0, \tau)\}^s = \vartheta^s,$$

where $q = e^{\pi i \tau}$ and $\Im(\tau) > 0$; and I consider the behavior of this function when q tends radially to a "rational point" $e^{2h\pi i/k}$ upon the unit circle. We may suppose that $h = 0, k = 1$, or that k is greater than unity and h positive, less than k, and prime to k.

If (2.11)

$$q = \boldsymbol{q} e^{2h\pi i/k},$$

so that $0 \leqq \boldsymbol{q} < 1$, $\boldsymbol{q} \to 1$, we have

$$\begin{aligned}\vartheta &= 1 + 2 \sum_{1}^{\infty} \boldsymbol{q}^{n2}\, e^{2n^2 h\pi i/k} \\ &= 1 + 2 \sum_{j=1}^{k} \sum_{l=0}^{\infty} \boldsymbol{q}^{(lk+j)^2}\, e^{2(lk+j)^2 h\pi i/k} \\ &= 1 + 2 \sum_{j=1}^{k} e^{2j^2 h\pi i/k} \sum_{l=0}^{\infty} \boldsymbol{q}^{(lk+j)^2}.\end{aligned}$$

Now

$$\sum_{k=0}^{\infty} \boldsymbol{q}^{(lk+j)^2} \sim \frac{\sqrt{\pi}}{2k}\left(\log \frac{1}{\boldsymbol{q}}\right)^{-\frac{1}{2}},$$

when $\boldsymbol{q} \to 1$. It follows that

$$(2.12)\quad \vartheta \sim \sqrt{\pi}\, \frac{S_{h,\,k}}{k}\left(\log \frac{1}{\boldsymbol{q}}\right)^{-\frac{1}{2}},$$

where

$$(2.121)\quad S_{h,\,k} = \sum_{j=1}^{k} e^{2j^2 h\pi i/k},$$

and

$$(2.13)\quad f(q) \sim \pi^{\frac{1}{2}s}\left(\frac{S_{h,\,k}}{k}\right)^s \left(\log \frac{1}{\boldsymbol{q}}\right)^{-\frac{1}{2}s},$$

it being understood that, when $S_{h,\,k} = 0$ (as is the case if, and only if, k is of the form $4m + 2$), this equation is to be understood as meaning

$$(2.131)\quad f(q) = o\left(\log \frac{1}{\boldsymbol{q}}\right)^{-\frac{1}{2}s}$$

2.2. The principle of the method is to write down a power-series

(2.21) $$f_{h,k}(q) = \sum c_{h,k,n}\, q^n,$$

which (a) is as simple and natural as possible and (b) behaves as much like $f(q)$ as possible when $q \to e^{2h\pi i/k}$; and to endeavor to approximate to the coefficients in $f(q)$ by means of the sums

(2.22) $$\rho_s(n) = \sum_{h,k} c_{h,k,n}.$$

It is plain that, in forming these sums, we may ignore values of k of the form $4m+2$.

The appropriate auxiliary function (2.21) is

(2.23) $$f_{h,k}(q) = \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}\left(\frac{S_{h,k}}{k}\right)^s F_s(q),$$

where

(2.231) $$F_s(q) = \sum_1^{\infty} n^{\frac{1}{2}s-1} q^n.$$

It is in fact well known that

$$F_s(x) - \Gamma(\tfrac{1}{2}s)\left(\log\frac{1}{x}\right)^{-\frac{1}{2}s}$$

is regular at $x = 1$.[7] We are thus led to take

(2.24) $$c_{h,k,n} = \frac{\pi^{\frac{1}{2}s} n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)}\left(\frac{S_{h,k}}{k}\right)^s e^{-2nh\pi i/k}$$

and

(2.25) $$\rho_s(n) = \sum_{h,k} c_{h,k,n} = \frac{\pi^{\frac{1}{2}s} n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)}\sum_1^{\infty} A_k,$$

where

(2.251) $$A_1 = 1, \qquad A_k = k^{-s}\sum_h (S_{h,k})^s e^{-2nh\pi i/k},$$

the summation extending over all positive values of h less than and prime to k. I call the series

(2.26) $$\rho_s(n) = \frac{\pi^{\frac{1}{2}s} n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)}\sum A_k = \frac{\pi^{\frac{1}{2}s} n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)} S$$

the *singular series*. The process by which it has been constructed is of a purely formal character. It remains (1) to investigate more rigorously its bearing on the solution of our problem and (2) to find its sum.

3. Proof that the sum of the singular series, when $s = 8$ or $s = 5$, is the number of representations of n

3.1. *Proof that* $\rho_8(n) = r_8(n)$.

3.11. When s is 3, 4, 5, 6, 7, or 8 (but not 2 or any number greater

[7] See, for example, E. Lindelöf, *Le calcul des résidus*, p. 139.

than 8) the sum of the singular series gives exactly the number of representations of n. In this section I prove this when $s = 8$ and when $s = 5$. These cases are perfectly typical, but formally a little simpler than the others.

Suppose first that $s = 8$. Then

$$\Theta_8(q) = 1 + \sum f_{h,k}(q) = 1 + \frac{\pi^4}{6}\sum\left(\frac{S_{h,k}}{k}\right)^8 F_8(qe^{-2h\pi i/k}). \tag{3.111}$$

Now

$$S_{h,k}^8 = \eta_k k^4,$$

where η_k is 1, 0, or 16 according as k is odd, oddly even, or evenly even. Also, if $x = e^{-y}$, we have

$$F_8(x) = \sum n^3 x^n = \sum n^3 e^{-ny} = \frac{d^2}{dy^2}(\tfrac{1}{4}\operatorname{cosech}^2 \tfrac{1}{2}y) = 6\sum\frac{1}{(y + 2n\pi i)^4},$$

where n runs through all integral values. Hence

$$F_8(qe^{-2h\pi i/k}) = \frac{6k^4}{\pi^4}\sum_n \frac{1}{\{2(nk+h) - k\tau\}^4},$$

and

$$\Theta_8(q) = 1 + \sum_{h,k,n}\frac{\eta_k}{\{2(nk+h) - k\tau\}^4}, \tag{3.112}$$

the summation extending over the values of h, k, and n already specified. If $k > 1$, $nk + h$ assumes all values prime to k; if $k = 1$, all values. Thus

$$\Theta_8(q) = 1 + \sum\frac{\eta_k}{(2h - k\tau)^4}; \tag{3.113}$$

where now $k = 1, 2, 3, \cdots$ and h assumes all values such that (h, k), the highest common factor of h and k, is unity. But this equation may be written

$$\begin{aligned}\Theta_8(q) &= 1 + \sum_{k=1,3,\ldots;\,(h,k)=1}\frac{1}{(2h-k\tau)^4} + \sum_{k=4,8,\ldots;\,(h,k)=1}\frac{16}{(2h-k\tau)^4} \\ &= 1 + \sum_{k=1,3,\ldots;\,(h,k)=1}\frac{1}{(2h-k\tau)^4} + \sum_{k=2,4,\ldots;\,(h,k)=1}\frac{1}{(h-k\tau)^4} \\ &= 1 + \sum\frac{1}{(h-k\tau)^4},\end{aligned} \tag{3.114}$$

where now $k = 1, 2, 3, \cdots$ and h assumes all values *of opposite parity to and prime to* k.

Multiplying both sides of (3.114) by

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots,$$

we obtain

$$\frac{\pi^4}{96}\Theta_8(q) = \frac{\pi^4}{96} + \sum\frac{1}{(h-k\tau)^4} \tag{3.1151}$$

or

(3.1152) $$\frac{\pi^4}{96}\Theta_8(q) = \sum \frac{1}{(h - k\tau)^4}.$$

In (3.1151) $k = 1, 2, 3, \cdots$, and in (3.1152) $k = 0, 1, 2, \cdots$; in each equation h assumes *all* values of opposite parity to k.

3.12. We now write

(3.121) $$\frac{\pi^4}{96}\Theta_8(q) = \chi(\tau),$$

and consider the effect on $\chi(\tau)$ of the modular substitutions

(3.1221) $$\tau' = \tau \pm 2,$$ (3.1222) $$\tau' = -1/\tau.$$

It is obvious in the first place, from (3.1151) or (3.1152), that

(3.123) $$\chi(\tau \pm 2) = \chi(\tau).$$

Again, we may write (3.1151) in the form

$$\chi(\tau) = \frac{\pi^4}{96} + \frac{1}{2}\sum_k{}' \sum_h \frac{1}{(h - k\tau)^4},$$

where h and k assume all values of opposite parity except that (as is indicated by the dash) the value $k = 0$ is omitted. Thus

$$\chi\left(-\frac{1}{\tau}\right) = \frac{\pi^4}{96} + \frac{1}{2}\tau^4 \sum_k{}' \sum_h \frac{1}{(h\tau + k)^4} = \frac{\pi^4\tau^4}{96} + \frac{1}{2}\tau^4 \sum_k \sum_h{}' \frac{1}{(h\tau + k)^4}.$$

Changing h and k into $-k$ and h, we obtain

(3.124) $$\chi(-1/\tau) = \tau^4 \chi(\tau).$$

Now $\{\vartheta_3(0, \tau)\}^8 = \vartheta^8 = \psi(\tau)$ satisfies the equations

$$\psi(\tau \pm 2) = \psi(\tau), \qquad \psi(-1/\tau) = \tau^4 \psi(\tau);$$

and so it follows, from (3.123) and (3.124), that the function

(3.125) $$\eta(\tau) = \chi(\tau)/\vartheta^8 = \chi(\tau)/\psi(\tau)$$

is invariant for the substitutions (3.122), and therefore for the modular subgroup which they generate, the group called by Klein-Fricke and Mordell Γ_3.

3.13. The next step in the proof is to show that $\eta(\tau)$ is bounded throughout the "fundamental polygon" G_3 associated with the group Γ_3. This region is defined by

$$\tau = x + iy, \qquad |\tau| \geqq 1, \qquad -1 \leqq x \leqq 1,$$

and has only the points $\tau = \pm 1$ in common with the real axis. It is therefore sufficient to show that $\eta(\tau)$ is bounded when τ approaches one or other of these points, say $\tau = 1$. For this purpose, following Mordell, I consider the effect of the substitution

$$\tau = 1 - \frac{1}{T}.$$

If we write $T = X + iY$, and suppose that $\tau \to 1$ from inside G_3, then $Y \to \infty$ and $|Q| = |e^{\pi i T}|$ is small. And

$$\chi\left(1 - \frac{1}{T}\right) = \frac{\pi^4}{96} + \frac{1}{2} T^4 \sum_k{}' \sum_h \frac{1}{\{k + (h-k)T\}^4} \tag{3.131}$$

$$= \frac{\pi^4}{96} + \frac{1}{2} T^4 \sum_k{}' \sum_h \frac{1}{(k + hT)^4}$$

$$= \frac{1}{2} T^4 \sum_k \sum_h \frac{1}{(k + hT)^4},$$

where now k assumes all integral values and h all odd values.

Write $hT = a$, $e^{\pi i a} = \zeta$, and sum with respect to k. We have

$$\sum_k \frac{1}{(k+a)^4} = -\frac{2}{3}\pi^4 \operatorname{cosec}^2 a\pi + \pi^4 \operatorname{cosec}^4 a\pi;$$

and this function, when expanded in powers of ζ, begins with the term

$$\tfrac{8}{3}\pi^4 \zeta^2 = \tfrac{8}{3}\pi^4 Q^{2h}.$$

Hence, when $T = X + iY$ and Y is large, we have

$$\chi\left(1 - \frac{1}{T}\right) = \frac{8}{3}\pi^4 T^4 Q^2 + \cdots,$$

$$\Theta_8(q) = 256 T^4 Q^2 + \cdots. \tag{3.132}$$

But we have also

$$\left\{\vartheta_3\left(0, 1 - \frac{1}{T}\right)\right\}^8 = T^4\{\vartheta_2(0, T)\}^8 = 256 T^4 Q^2 + \cdots, \tag{3.133}$$

and so

$$\eta(\tau) = \eta\left(1 - \frac{1}{T}\right) \to 1. \tag{3.134}$$

Thus $\eta(\tau)$ is an invariant of Γ_3 and bounded throughout G_3; and is therefore necessarily a constant, which is plainly unity.

It follows that

$$\vartheta^8 = \Theta_8(q) \tag{3.135}$$

and so that

$$\rho_8(n) = r_8(n). \tag{3.136}$$

3.2. *Proof that* $\rho_5(n) = r_5(n)$.

3.21. When $s = 5$ the proof proceeds on the same lines, but is not quite so simple. We shall require certain well-known identities which I state as lemmas.

LEMMA 3.211.[8] *If h and k are positive integers of opposite parity, then*

$$(3.2111)\qquad \sum_{j=1}^{k} e^{j^2 h\pi i/k} = \sqrt{i}\sqrt{\frac{k}{h}}\sum_{j=1}^{h} e^{-j^2 k\pi i/h};$$

and if h and k are positive integers, and h odd, then

$$(3.2112)\qquad \sum_{j=1}^{k} (-1)^j e^{j^2 h\pi i/k} = \sqrt{i}\sqrt{\frac{k}{h}}\sum_{j=1}^{h} e^{-(j-\frac{1}{2})^2 k\pi i/h}.$$

Here $\sqrt{i} = e^{\frac{1}{4}\pi i}$.

LEMMA 3.212. *Suppose that* $0 < \nu < 1$, *and that* σ *and the real part of t are positive. Then*

$$(3.2121)\qquad \frac{(2\pi)^\sigma}{\Gamma(\sigma)}\sum_{0}^{\infty}(m+\nu)^{\sigma-1}e^{-2\pi t(m+\nu)} = \sum_{-\infty}^{\infty}\frac{e^{2n\nu\pi i}}{(t+ni)^\sigma},$$

where

$$(t+ni)^\sigma = exp\{\sigma\log(t+ni)\} = exp(\sigma\log|t+ni| + \sigma\phi i)$$

and $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$. *The formula still holds for* $\nu = 0$, *if* $\sigma > 1$.

This result is due to Lipschitz.[9] We shall require two special cases.

(i) Suppose that $\sigma = \frac{1}{2}s > 1$, $\nu = 0$, $t = -\frac{1}{2}i\tau$, $x = e^{\pi i\tau}$; so that $\Im(\tau) > 0$ and $|x| < 1$. Then we obtain

$$(3.2122)\qquad \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}F_s(x) = \sum_{-\infty}^{\infty}\frac{1}{\{(2n-\tau)i\}^{\frac{1}{2}s}},$$

where

$$(3.21221)\qquad \{(2n-\tau)i\}^{\frac{1}{2}s} = exp\{\tfrac{1}{2}s\log|(2n-\tau)i| + \tfrac{1}{2}s\phi i\}$$

and $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$.

(ii) Suppose that $\nu = \frac{1}{2}(1+\theta)$, $\theta = \lambda/K$, where K and λ are integers and $-K < \lambda < K$, and $t = -K\tau i$. Then

$$(3.2123)\qquad \sum_{-\infty}^{\infty}\frac{(-1)^n e^{n\theta\pi i}}{\{(n-K\tau)i\}^\sigma} = x^{K+\lambda}P(x),$$

where $P(x)$ is an ascending power series in x.

3.22. Supposing now that $s = 5$, we have

$$(3.221)\qquad \Theta_5(q) = 1 + \frac{4\pi^2}{3}\sum\left(\frac{S_{h,k}}{k}\right)^5 F_5(qe^{-2h\pi i/k}),$$

[8] See, for example, G. Landsberg, *Zur Theorie der Gaussschen Summen und der linearen Transformation der Thetafunktionen*, Journal für Mathematik,, vol. 111 (1893), pp. 234–253. Both formulas are included in formula (17*b*), p. 243, of Landsberg's memoir. The first is also proved by Lindelöf, loc. cit., pp. 73–75.

[9] R. Lipschitz, *Untersuchung der Eigenschaften einer Gattung von unendlichen Reihen*, Journal für Mathematik, vol. 105 (1889), pp. 127–156.

and

$$S_{h,k}^4 = \eta_k k^2,$$

where now η_k is 1, 0, or -4 according as k is odd, oddly even, or evenly even. Thus

$$\Theta_5(q) = 1 + \frac{4\pi^2}{3} \sum \frac{\eta_k}{k^3} S_{h,k} F_5(qe^{-2h\pi i/k}). \tag{3.222}$$

Substituting from (3.2122), we obtain

$$\Theta_5(q) = 1 + \sum \frac{\eta_k}{\sqrt{k}} \frac{S_{h,k}}{[\{2(nk+h) - k\tau\}i]^{5/2}}, \tag{3.223}$$

or

$$\Theta_5(q) = 1 + \sum \frac{\eta_k}{\sqrt{k}} \frac{S_{h,k}}{\{(2h - k\tau)i\}^{5/2}}, \tag{3.224}$$

the ranges of summation in these equations being the same as in (3.112) and (3.113) respectively. The last equation can be expressed in a more convenient form by introducing the sum

$$T_{h,k} = \sum_0^{k-1} e^{j^2 h\pi i/k}. \tag{3.225}$$

In fact (3 224) may be written

$$\Theta_5(q) = 1 + \sum_{k=1,3,\cdots;\,(h,k)=1} \frac{1}{\sqrt{k}} \frac{S_{h,k}}{\{(2h-k\tau)i\}^{5/2}} - \sum_{k=4,8,\cdots;\,(h,k)=1} \frac{4}{\sqrt{k}} \frac{S_{h,k}}{\{(2h-k\tau)i\}^{5/2}}.$$

In the first sum k is odd; $2h = H$ is even and prime to k; and $S_{h,k} = T_{H,k}$. In the second $k = 2K$, where K runs through all even values; h is odd and prime to K; and $S_{h,k} = 2T_{h,K}$. Effecting these substitutions, and then replacing H (or K) by h (or k), we obtain

$$\Theta_5(q) = 1 + \sum \frac{(-1)^h}{\sqrt{k}} \frac{T_{h,k}}{\{(h - k\tau)i\}^{5/2}},$$

where now $k = 1, 2, 3, \cdots$ and h assumes all values of opposite parity and prime to k.[10]

Multiplying by

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots,$$

and observing that, if λ is odd,

$$(-1)^{\lambda h} = (-1)^h, \qquad \sqrt{\lambda k}\{(\lambda h - \lambda k\tau)i\}^{5/2} = \lambda^3 \sqrt{k}\{(h - k\tau)i\}^{5/2},$$

$$T_{\lambda h, \lambda k} = \lambda T_{h,k},$$

[10] This is equation (8) of my former paper.

we obtain

$$(3.226)\qquad \frac{\pi^2}{8}\Theta_5(q) = \frac{\pi^2}{8} + \sum \frac{(-1)^h}{\sqrt{k}} \frac{T_{h,k}}{\{(h-k\tau)i\}^{5/2}},$$

where h now assumes all values of opposite parity to k.[11]

3.23. The discussion now follows the lines of 3.12 and 3.13. We write

$$(3.231)\qquad \frac{\pi^2}{8}\Theta_5(q) = \chi(\tau),$$

and it is obvious at once that

$$(3.232)\qquad \chi(\tau \pm 2) = \chi(\tau).$$

The discussion of the transformation $\tau' = -1/\tau$ requires a little more care, owing to the presence of many-valued functions under the sign of summation. It is convenient to begin by including negative values of k.

We write generally

$$z^s = exp\{s \log |z| + i \operatorname{am} z\}$$

where the particular value of $\operatorname{am} z$ to be selected has to be fixed by special convention. Thus in $\{(h-k\tau)i\}^{5/2}$, where $k>0$, $\operatorname{am}\{(h-k\tau)i\}$ lies (as has already been explained) between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. We now agree that, if k is still positive, $\operatorname{am}(-k) = \pi$, so that $\sqrt{-k} = i\sqrt{k}$, while $\operatorname{am}\{(-h+k\tau)i\}$ lies between $-\frac{3}{2}\pi$ and $-\frac{1}{2}\pi$. It will easily be verified that

$$(3.233)\qquad \sqrt{-k}\{(-h+k\tau)i\}^{5/2} = \sqrt{k}\{(h-k\tau)i\}^{5/2}.$$

Further, we write by definition

$$(3.234)\qquad T_{-h,-k} = T_{h,k}.$$

We know from (3.2111) that, when h and k are both positive,

$$T_{h,k} = \sqrt{i}\sqrt{\frac{k}{h}}\,T_{-k,h};$$

and it is easy to verify that, with our conventions, we have generally

$$(3.235)\qquad T_{h,k} = \epsilon\sqrt{i}\sqrt{\frac{k}{h}}\,T_{-k,h},$$

where $\epsilon = 1$ unless $h>0$, $k<0$, in which case $\epsilon = -1$.

3.24. We have, from (3.226), (3.233), and (3.234),

$$(3.241)\qquad \chi(\tau) = \frac{\pi^2}{8} + \frac{1}{2}\sum_k{}'\sum_h \frac{(-1)^h}{\sqrt{k}} \frac{T_{h,k}}{\{(h-k\tau)i\}^{5/2}}$$

$$= \frac{\pi^2}{8} + \frac{\pi^2}{8}(-\tau i)^{-5/2} + \frac{1}{2}\sum{}' \frac{(-1)^h}{\sqrt{k}} \frac{T_{h,k}}{\{(h-k\tau)i\}^{5/2}},$$

[11] This equation takes the place of (9) of my former paper, which is not printed correctly. The first term on the right is omitted, and $k=0$ is included wrongly under the sign of summation.

where now h and k are any integers other than zero and of opposite parity. Writing $-1/\tau$ for τ in (3.241), using (3.235), and then replacing h and k by K and $-H$, we obtain

$$(3.242)\quad \chi\left(-\frac{1}{\tau}\right)=\frac{\pi^2}{8}+\frac{\pi^2}{8}\left(\frac{i}{\tau}\right)^{-5/2}+\frac{1}{2}\sqrt{i}\sum{}'\frac{(-1)^K\epsilon}{\sqrt{K}}\frac{T_{H,K}}{\{(K-H/\tau)i\}^{5/2}},$$

where ϵ is 1 unless H and K are both positive, and then -1, and

$$\operatorname{am}\{(K-H/\tau)i\}$$

lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ if $H<0$, between $-\frac{3}{2}\pi$ and $-\frac{1}{2}\pi$ if $H>0$.

It may be verified without difficulty that

$$(3.243)\qquad \left\{\left(K-\frac{H}{\tau}\right)i\right\}^{5/2}=\epsilon\left(-\frac{1}{\tau}\right)^{5/2}\{(H-K\tau)i\}^{5/2},$$

where $0<\operatorname{am}(-1/\tau)<\pi$ and the value of $\operatorname{am}\{(H-K\tau)i\}$ is fixed in accordance with our previous conventions. Consider, for example, the case $H>0,\ K<0$. In this case

$$-\tfrac{3}{2}\pi<\alpha=\operatorname{am}\{(K-H/\tau)i\}<-\tfrac{1}{2}\pi,$$

$$0<\beta=\operatorname{am}(-1/\tau)<\pi,$$

and

$$-\tfrac{1}{2}\pi<\gamma=\operatorname{am}\{(H-K\tau)i\}<\tfrac{1}{2}\pi.$$

Thus $\beta+\gamma$ lies between $-\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, and, as α differs from $\beta+\gamma$ by a multiple of 2π, we must have $\alpha=\beta+\gamma-2\pi$ and

$$\left\{\left(K-\frac{H}{\tau}\right)i\right\}^{5/2}=e^{-5\pi i}\left(-\frac{1}{\tau}\right)^{5/2}\{(H-K\tau)i\}^{5/2}$$
$$=-\left(-\frac{1}{\tau}\right)^{5/2}\{(H-K\tau)i\}^{5/2},$$

in agreement with (3.243). The other possible cases may be treated similarly.

Thus

$$(3.244)\quad \chi\left(-\frac{1}{\tau}\right)=\frac{\pi^2}{8}+\frac{\pi^2}{8}\left(\frac{i}{\tau}\right)^{-5/2}$$
$$+\tfrac{1}{2}\sqrt{i}\left(-\frac{1}{\tau}\right)^{-5/2}\sum{}'\frac{(-1)^K}{\sqrt{K}}\frac{T_{H,K}}{\{(H-K\tau)i\}^{5/2}},$$

where $-\frac{1}{2}\pi<\operatorname{am}(i/\tau)<\frac{1}{2}\pi,\ 0<\operatorname{am}(-1/\tau)<\pi$. And this equation leads to

$$(3.245)\quad \chi\left(-\frac{1}{\tau}\right)=\frac{\pi^2}{8}\left(\frac{i}{\tau}\right)^{-5/2}+\tfrac{1}{2}\sqrt{i}\left(-\frac{1}{\tau}\right)^{-5/2}\sum_K{}'\sum_H\frac{(-1)^K}{\sqrt{K}}\frac{T_{H,K}}{\{(H-K\tau)i\}^{5/2}}$$
$$=\left(\frac{\tau}{i}\right)^{5/2}\left[\frac{\pi^2}{8}+\frac{1}{2}\sum_K{}'\sum_H\frac{(-1)^H}{\sqrt{K}}\frac{T_{H,K}}{\{(H-K\tau)i\}^{5/2}}\right]$$
$$=\left(\frac{\tau}{i}\right)^{5/2}\chi(\tau).$$[12]

This is the same functional equation as is satisfied by ϑ^5. Hence

$$(3.246)\qquad \eta(\tau)=\chi(\tau)/\vartheta^5$$

is an invariant for each of the substitutions (3.122), and so for Γ_3.

3.25. It remains to verify that $\eta(\tau)$ is bounded in G_3. As in 3.13, it is only necessary to consider the neighborhood of $\tau = 1$. Putting $\tau = 1 - 1/T$, as in 3.13, in (3.241), and then writing $h = H - K$, $k = H$, we obtain

$$(3.251)\quad \frac{\pi^2}{8}\chi\left(1-\frac{1}{T}\right)=\frac{\pi^2}{8}+\frac{1}{2}\sum_H{}'\sum_K\frac{(-1)^H}{\sqrt{H}}\frac{T_{H-K,H}}{\{(-K+H/\tau)i\}^{5/2}},$$

where H assumes all values save 0 and K all odd values, and

$$\mathrm{am}\{(-K+H/\tau)i\}$$

lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, or between $-\frac{3}{2}\pi$ and $-\frac{1}{2}\pi$, according as $H > 0$ or $H < 0$.

Now

$$(3.252)\qquad T_{H-K,H}=\sum_0^{H-1}(-1)^j e^{-j^2K\pi i/H}=U_{K,H},$$

say. It follows from (3.2112) that, if h and k are both positive, and h is odd, and

$$(3.253)\qquad U_{h,k}=\sum_0^{k-1}(-1)^j e^{-j^2h\pi i/k},\qquad W_{h,k}=\sum_1^k e^{(j-\frac{1}{2})^2h\pi i/k},$$

then

$$U_{h,k}=\sqrt{-i}\sqrt{\frac{k}{h}}\,W_{k,h};$$

and it is easily verified that, if we adopt the same conventions as in 3.23 concerning the meanings of $\sqrt{-h}$, $\sqrt{-k}$, $U_{-h,-k}$, and $W_{-h,-k}$, we have

[12] We first restore the terms for which $H = 0$, and then observe that

$$(-1)^H=-(-1)^K.$$

We have to verify that, with our conventions,

$$-\sqrt{i}(-1/\tau)^{-\frac{5}{2}}(-\tau i)^{-\frac{5}{2}}=1$$

and

$$-\sqrt{i}(-1/\tau)^{-\frac{5}{2}}=-(i/\tau)^{-\frac{5}{2}}:$$

these verifications present no difficulty.

generally

$$U_{h,k} = \epsilon\sqrt{-i}\sqrt{\frac{k}{h}}W_{k,h}, \tag{3.254}$$

where $\epsilon = 1$ unless $h < 0$, $k > 0$, in which case $\epsilon = -1$.

Using this equation in (3.251), we obtain

$$\frac{\pi^2}{8}\chi\left(1-\frac{1}{T}\right) = \frac{\pi^2}{8} - \tfrac{1}{2}\sqrt{-i}\sum_H{}'\sum_K \frac{(-1)^H\epsilon}{\sqrt{K}}\frac{W_{H,K}}{\{(-K+H/T)i\}^{5/2}} \tag{3.255}$$

$$= -\tfrac{1}{2}\sqrt{-i}\sum_H\sum_K \frac{(-1)^H\epsilon}{\sqrt{K}}\frac{W_{H,K}}{\{(-K+H/T)i\}^{5/2}}.$$

It is now easy to verify, by arguments similar to those of 3.24, that

$$\{(-K+H/T)i\}^{5/2} = \epsilon T^{-5/2}\{(H-KT)i\}^{5/2},$$

where $\operatorname{am} T$ lies between 0 and π, while $\operatorname{am}\{(H-KT)i\}$ obeys our previous conventions. We thus obtain

$$\frac{\pi^2}{8}\chi\left(1-\frac{1}{T}\right) = -\tfrac{1}{2}\sqrt{-i}\,T^{5/2}\sum_H\sum_K \frac{(-1)^H}{\sqrt{K}}\frac{W_{H,K}}{\{(H-KT)i\}^{5/2}},$$

the summation being now limited only by the fact that K is odd. In virtue of (3.233), this equation may be written

$$\frac{\pi^2}{8}\chi\left(1-\frac{1}{T}\right) = -\sqrt{-i}\,T^{5/2}\sum_{K=1,3,5,\ldots}\frac{1}{\sqrt{K}}\sum_H \frac{(-1)^H W_{H,K}}{\{(H-KT)i\}^{5/2}}. \tag{3.256}$$

3.26. The series in (3.256) may be expressed in the form

$$\sum_{K=1,3,5,\ldots}\frac{1}{\sqrt{K}}\sum_{j=1}^{K}\sum_H \frac{(-1)^H e^{-(j-\frac{1}{2})^2 H\pi i/K}}{\{(H-KT)i\}^{5/2}}.$$

Suppose that

$$(j-\tfrac{1}{2})^2 \equiv \lambda_j \pmod{2K} \qquad (-K < \lambda_i < K),$$

and $\theta_j = \lambda_j/K$. Then

$$\sum_H \frac{(-1)^H e^{-(j-\frac{1}{2})^2 H\pi i/K}}{\{(H-KT)i\}^{5/2}} = \sum_H \frac{(-1)^H e^{-H\theta_j\pi i}}{\{(H-KT)i\}^{5/2}}$$

may, by (3.2123), be expanded as a power series in $Q = e^{\pi i T}$, in which the lowest power of Q is

$$Q^{K+\lambda_j}.$$

The smallest possible values of $K+\lambda_j$ are $\frac{1}{4}, \frac{5}{4}, \cdots$; and $K+\lambda_j = \frac{1}{4}$ involves

$$j^2 - j + \tfrac{1}{4} \equiv -K + \tfrac{1}{4} \pmod{2K}$$

or

$$j^2 - j = (2p-1)K,$$

where p is an integer, i.e. an equation whose left-hand side is even and whose right-hand side is odd. Thus $K + \lambda_j \geqq \frac{5}{4}$; and the left-hand side of (3.256) is the product of $T^{5/2} Q^{5/4}$ by an ascending power-series in Q. But

$$\vartheta_3\left(0, 1 - \frac{1}{T}\right) = \frac{\sqrt{T}}{\sqrt{i}} \vartheta_2(0, T)$$

is the product of $T^{1/2} Q^{1/4}$ by a power-series in Q. It now follows, just as in 3.13, that $\eta(\tau)$ is bounded, and so is a constant, which is plainly unity.

We have thus established the identity of Θ_s and ϑ^s, and so of $\rho_s(n)$ and $r_s(n)$, when $s = 8$ and $s = 5$. The same method may be used for any value of s from 5 to 8 inclusive.[13] In order to complete the solution of our problem,.we have to sum the singular series (2.26).

4. General rules for the summation of the singular series

4.1. *The value of* $S_{h,k}$.

The known results concerning the value of the Gaussian sum $S_{h,k}$ are as follows.[14] We assume that $(h, k) = 1$.

If k and k' are prime to one another

$$S_{h,kk'} = S_{hk',k} S_{hk,k'}. \tag{4.11}$$

We need therefore consider only the cases in which $k = 2^\lambda$ or $K = p^\lambda$, p being an odd prime.

If $k = 2$

$$S_{h,k} = 0. \tag{4.121}$$

If $k = 2^\lambda = 2^{2\mu+1}$, and $\mu > 0$,

$$S_{h,k} = 2^{\mu+1} e^{\frac{1}{4}h\pi i}. \tag{4.122}$$

If $k = 2^\lambda = 2^{2\mu}$, and $\mu > 0$,

$$S_{h,k} = 2^\mu (1 + i^h) = 2^{\mu+1} \cos \tfrac{1}{4}h\pi\, e^{\frac{1}{4}h\pi i}. \tag{4.123}$$

If $k = p$,

$$S_{h,k} = \left(\frac{h}{p}\right) i^{\frac{1}{4}(p-1)^2}\sqrt{p}, \tag{4.131}$$

where (h/p) is the well known symbol of Legendre and Jacobi.

---

[13] When $s = 2$ or $s > 8$, the conclusion is false. The cases $s = 3$ and $s = 4$ are exceptional. The conclusion is true, but new difficulties arise in the proof because the series used are not all absolutely convergent. These difficulties are easily surmounted when $s = 4$, but are more serious when $s = 3$.

[14] For proofs of these assertions see the chapter on Gauss's sums in the second volume of Bachmann's *Zahlentheorie*. A less complete account is given in Dirichlet-Dedekind, *Vorlesungen über Zahlentheorie*, ed. 4, 1894, pp. 287 et seq.

If $k = p^\lambda = p^{2\mu+1}$, and $\mu > 0$,

(4.132) $$S_{h,\,k} = p^\mu S_{h,\,p}.$$

Finally, if $k = p^\lambda = p^{2\mu}$, and $\mu > 0$,

(4.133) $$S_{h,\,k} = p^\mu.$$

These formulas enable us to write down the value of $S_{h,\,k}$ for all co-prime pairs of values of h and k.

The multiplication rule for A_k.

4.2. The first step is to prove that

(4.21) $$A_{kk'} = A_k A_{k'}$$

whenever k and k' are prime to one another.

In the formula which defines A_k, viz.

$$k^s A_k = \sum (S_{h,\,k})^s e^{-2nh\pi i/k},$$

h assumes all positive values less than and prime to k. Let us call this set of values, or any set congruent to this set to modulus k, a *k-set.* It is easy to see that if h runs through a k-set, and h' through a k'-set, then

$$\boldsymbol{h} = hk' + h'k$$

runs through a kk'-set. For the number of values of $\boldsymbol{h}$ is

$$\phi(k)\phi(k') = \phi(kk'),$$

and it is obvious that all are prime to kk' and incongruent to modulus kk'.

Thus

$$\begin{aligned}(kk')^s A_{kk'} &= \sum_{\boldsymbol{h}} (S_{\boldsymbol{h},\,kk'})^s e^{-2n\boldsymbol{h}\pi i/kk'} \\ &= \sum_{\boldsymbol{h}} (S_{\boldsymbol{h}k',\,k})^s (S_{\boldsymbol{h}k,\,k'})^s e^{-2n\boldsymbol{h}\pi i/kk'},\end{aligned}$$

by (4.11). But

$$S_{\boldsymbol{h}k',\,k} = \sum_{j=1}^{k} e^{2j^2hk'^2\pi i/k} = \sum_{j=1}^{k} e^{2(jk')^2h\pi i/k} = S_{h,\,k},$$

since jk' runs through a complete system of residues to modulus k; and similarly $S_{\boldsymbol{h}k,\,k'} = S_{h',\,k'}$. Thus

$$\begin{aligned}(kk')^s A_{kk'} &= \sum_{h,\,h'} (S_{h,\,k})^s (S_{h',\,k'})^s e^{-2n\boldsymbol{h}\pi i/kk'} \\ &= \sum_{h} (S_{h,\,k})^s e^{-2nh\pi i/k} \sum_{h'} (S_{h',\,k'})^s e^{-2nh'\pi i/k'} \\ &= (kk')^s A_k A_{k'};\end{aligned}$$

which proves (4.21).

It follows[15] that

(4.22) $$S = A_1 + A_2 + A_3 + \cdots = 1 + A_2 + A_3 + \cdots = \prod \chi_p$$

where

(4.23) $$\chi_p = 1 + A_p + A_{p^2} + A_{p^3} + \cdots$$

and p runs through all prime values.

Calculation of A_{2^λ}.

4.3. Suppose first that $p = 2$. Then the value of A_{2^λ} is given by the following system of rules.

4.31. $$A_2 = 0.$$

4.32. *If λ is odd and greater than 1,*

(4.321) $$A_{2^\lambda} = 0,$$

unless

(4.3221) $$n \equiv 0 \pmod{2^{\lambda-3}}$$

and

(4.3222) $$\nu - s = 2^{-(\lambda-3)} n - s \equiv 0 \pmod 4,$$

in which case

(4.3223) $$A_{2^\lambda} = 2^{-(\frac{1}{2}s-1)(\lambda-1)} e^{-\frac{1}{4}(\nu-s)\pi i}.$$

If $\lambda = 3$, (4.3221) is satisfied automatically.

Let $\lambda = 2\mu + 1$ $(\mu > 0)$. Then

$$S_{h, 2^\lambda} = 2^{\mu+1} e^{\frac{1}{4}h\pi i} = 2^{\mu-1} S_{h, 8},$$

by (4.122). We write

$$h = 8z + h' \qquad (z = 0, 1, \cdots, 2^{\lambda-3} - 1;\ h' = 1, 3, 5, 7).$$

Then

$$A_{2^\lambda} = 2^{-s(\mu+2)} \sum_h (S_{h, 8})^s e^{-2nh\pi i/2^{2\mu+1}}$$

$$= 2^{-s(\mu+2)} \sum_{h'} (S_{h', 8})^s e^{-2nh'\pi i/2^{2\mu+1}} \sum_z e^{-2nz\pi i/2^{2\mu-2}},$$

which vanishes (in virtue of the summation with respect to z) unless $n \equiv 0 \pmod{2^{2\mu-2}}$.

If $n = 2^{2\mu-2}\nu$, we have

$$A_{2^\lambda} = 2^{2\mu-2-s(\mu+2)} \sum_{h'} (S_{h', 8})^s e^{-\frac{1}{4}\nu h'\pi i}.$$

The sum with respect to h' is

$$2^{2s}\{e^{-\frac{1}{4}(\nu-s)\pi i} + e^{-\frac{3}{4}(\nu-s)\pi i} + e^{-\frac{5}{4}(\nu-s)\pi i} + e^{-\frac{7}{4}(\nu-s)\pi i}\},$$

[15] We assume that the series and product are absolutely convergent. This is obviously the case if $s > 4$, as $S_{h,k} = O(\sqrt{k})$, $A_k = O(k^{1-\frac{1}{2}s})$, and $1 - \frac{1}{2}s < -1$.

which is 0 or $2^{2s+2}e^{-\frac{1}{4}(\nu-s)\pi i}$ according as (4.3222) is not or is satisfied. This completes the proof of 4.32.

4.33. *If* λ *is even and greater than* 1,

(4.331) $$A_{2^\lambda}=0$$

unless

(4.3321) $$n\equiv 0\ (\mathrm{mod}\ 2^{\lambda-2}),$$

in which case

(4.3322) $$A_{2^\lambda}=2^{-(\frac{1}{2}s-1)(\lambda-1)}\cos\left(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}s\pi\right),$$

where $n=2^{\lambda-2}\nu$. *If* $\lambda=2$, *the last formula holds in any case.*

If $\lambda=2\mu$, we have

$$S_{h,2^\lambda}=2^{\mu+1}\cos\tfrac{1}{4}h\pi\, e^{\frac{1}{4}h\pi i}=2^{\mu-1}S_{h,4},$$

by (4.123). We write

$$h=4z+h'\qquad(z=0,1,\cdots,2^{\lambda-2}-1;\ h'=1,3).$$

Then

$$A_{2^\lambda}=2^{-s(\mu+1)}\sum_h (S_{h,4})^s e^{-2nh\pi i/2^{2\mu}}$$

$$=2^{-s(\mu+1)}\sum_{h'}(S_{h',4})^s e^{-2nh'\pi i/2^{2\mu}}\sum_z e^{-2nz\mu i/2^{2\mu-2}},$$

which vanishes (in virtue of the summation with respect to z) unless $n\equiv 0$ (mod $2^{2\mu-2}$). But if $n=2^{2\mu-2}\nu$, we have

$$A_{2^\lambda}=2^{2\mu-2-s(\mu+1)}\sum_{h'}(S_{h',4})^s e^{-\frac{1}{2}\nu h'\pi i},$$

and the sum here is

$$2^{2s}\{(\cos\tfrac{1}{4}\pi)^s e^{\frac{1}{4}s\pi i-\frac{1}{2}\nu\pi i}+(\cos\tfrac{3}{4}\pi)^s e^{\frac{3}{4}s\pi i-\frac{3}{2}\nu\pi i}\}=2^{\frac{3}{2}s+1}\cos\left(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}s\pi\right).$$

This completes the proof of 4.33. If $\lambda=2$, z disappears from the argument and h and h' are identical.

Calculation of A_{p^λ} *when* p *is odd.*

4.4. The corresponding results when p is odd are as follows.

4.41. *If* $n\not\equiv 0$ (mod p) *then*

(4.411) $$A_p=-p^{-\frac{1}{2}s}\qquad(s\equiv 0),$$

(4.412) $$A_p=\left(\frac{n}{p}\right)p^{-\frac{1}{2}(s-1)}\qquad(s\equiv 1),$$

(4.413) $$A_p=-(-1)^{\frac{1}{4}(p-1)}p^{-\frac{1}{2}s}\qquad(s\equiv 2),$$

(4.414) $$A_p=(-1)^{\frac{1}{4}(p-1)}\left(\frac{n}{p}\right)p^{-\frac{1}{2}(s-1)}\qquad(s\equiv 3);$$

the congruences for s *referring to modulus* 4. *But if* $n\equiv 0$ (mod p) *then*

(4.415) $$A_p=(p-1)p^{-\frac{1}{2}s}\qquad(s\equiv 0),$$

(4.416) $$A_p = 0 \qquad (s \equiv 1, 3),$$

(4.417) $$A_p = (-1)^{\frac{1}{2}(p-1)}(p-1)p^{-\frac{1}{2}s} \qquad (s \equiv 2).$$

We have

$$A_p = p^{-s}\sum_h (S_{h,p})^s e^{-2nh\pi i/p} = i^{\frac{1}{4}s(p-1)^2} p^{-\frac{1}{2}s} \sum_h \left(\frac{h}{p}\right)^s e^{-2nh\pi i/p}.$$

If s is even, this is

$$i^{\frac{1}{4}s(p-1)^2} p^{-\frac{1}{2}s} \sum_h e^{-2nh\pi i/p},$$

and the sum is equal to -1 or to $p-1$ according as n is not or is a multiple of p. This leads at once to the results stated for even values of s.

If on the other hand s is odd, we have

$$A_p = i^{\frac{1}{4}s(p-1)^2} p^{-\frac{1}{2}s} \sum_h \left(\frac{h}{p}\right) e^{-2nh\pi i/p},$$

which is equal to 0 if n is a multiple of p, and to

$$i^{\frac{1}{4}(s-1)(p-1)^2} \left(\frac{n}{p}\right) p^{-\frac{1}{2}(s-1)}$$

otherwise. We thus obtain the results stated for odd values of s.

4.42. *If λ is odd and greater than* 1, *then*

(4.421) $$A_{p^\lambda} = 0$$

if $n \not\equiv 0 \pmod{p^{\lambda-1}}$;

(4.422) $$A_{p^\lambda} = p^{-(\frac{1}{2}s-1)(\lambda-1)} A_p(\nu)$$

if $n = p^{\lambda-1}\nu$ *and* $\nu \not\equiv 0 \pmod p$; *and*

(4.4231) $$A_{p^\lambda} = (p-1)p^{\lambda-1-\frac{1}{2}s\lambda} \qquad (s \equiv 0),$$

(4.4232) $$A_{p^\lambda} = 0 \qquad (s \equiv 1, 3),$$

(4.4233) $$A_{p^\lambda} = (-1)^{\frac{1}{2}(p-1)}(p-1)p^{\lambda-1-\frac{1}{2}s\lambda} \qquad (s \equiv 2),$$

if $n \equiv 0 \pmod{p^\lambda}$.

If $\lambda = 2\mu + 1$, $\mu > 0$, we have $S_{h,p^\lambda} = p^\mu S_{h,p}$, by (4.132). We write

$$h = pz + h' \qquad (z = 0, 1, \cdots, p^{\lambda-1} - 1;\ h' = 1, 2, \cdots, p-1).$$

Then

$$A_{p^\lambda} = p^{-s(\mu+1)} \sum_h (S_{h,p})^s e^{-2nh\pi i/p^{2\mu+1}}$$
$$= p^{-s(\mu+1)} \sum_{h'} (S_{h',p})^s e^{-2nh'\pi i/p^{2\mu+1}} \sum_z e^{-2nz\pi i/p^{2\mu}}.$$

If $n \not\equiv 0 \pmod{p^{2\mu}}$, the sum with respect to z vanishes, and we obtain (4.421).

If $n = p^{2\mu}\nu$, we obtain

$$A_{p^\lambda} = p^{2\mu - s(\mu+1)} \sum_{h'} (S_{h', p})^s e^{-2\nu h'\pi i/p}.$$

If $\nu \not\equiv 0 \pmod{p}$, this is $p^{-(s-2)\mu} A_p(\nu)$, and we obtain (4.422). But if $\nu \equiv 0 \pmod{p}$ we have

$$\sum_{h'} (S_{h', p})^s = i^{\frac{1}{4}s(p-1)^2} p^{\frac{1}{2}s} \sum_{h'} \left(\frac{h'}{p}\right)^s,$$

and we obtain the equations (4.423).

4.43. *If λ is even and greater than* 1, *then*

(4.431) $$A_{p^\lambda} = 0$$

if $n \not\equiv 0 \pmod{p^{\lambda-1}}$;

(4.432) $$A_{p^\lambda} = -p^{\lambda-1-\frac{1}{2}s\lambda}$$

if $n = p^{\lambda-1}\nu$ *and* $\nu \not\equiv 0 \pmod{p}$; *and*

(4.433) $$A_{p^\lambda} = (p-1)p^{\lambda-1-\frac{1}{2}s\lambda}$$

if $n \equiv 0 \pmod{p^\lambda}$.

If $\lambda = 2\mu$, we have $S_{h, p^\lambda} = p^\mu$, by (4.133). Hence

$$A_{p^\lambda} = p^{-s\mu} \sum_{h'} e^{-2nh'\pi i/p^{2\mu}} \sum_{z} e^{-2nz\pi i/p^{2\mu-1}},$$

which is zero if $n \not\equiv 0 \pmod{p^{2\mu-1}}$, and

$$p^{2\mu-1-s\mu} \sum_{h'} e^{-2\nu h'\pi i/p}$$

if $n = p^{2\mu-1}\nu$; and the sum here is equal to -1 or $p-1$ according as ν is not or is divisible by p.

5. Summation of the singular series when $s = 8$

5. 1. The formulas of Section 4 enable us to sum the singular series whatever the value of s. I take as typical the cases $s = 8$ and $s = 5$. I suppose first that $s = 8$ and that n has no squared factor. We have to determine the factors χ_p of (4.22).

In the first place, let $p = 2$. Then, as n is not divisible by 4, we have $A_{16} = A_{32} = \cdots = 0$, by (4.321) and (4.331); and also $A_8 = 0$, by (4.321), since $\nu = 1$ and $\nu - s$ is not a multiple of 4. If n is odd, $A_4 = 0$, by (4.331); but if n is even,

$$A_4 = 2^{-3} \cos \tfrac{1}{2}\nu\pi = -\tfrac{1}{8},$$

by (4.3322). Finally $A_2 = 0$ in any case, by 4.31. Thus

(5.11) $$\chi_2 = 1\ (n\ odd), \qquad \chi_2 = \tfrac{7}{8}\ (n\ even).$$

Next, suppose p odd and $p \nmid n$.[16] Then $A_{p^2} = A_{p^3} = \cdots = 0$, by (4.421) and (4.431), and $A_p = -p^{-4}$, by (4.411). Thus

$$\chi_p = 1 - p^{-4} \qquad (p \nmid n). \tag{5.12}$$

Finally suppose p odd and $p|n$. Then $A_{p^3} = A_{p^4} = \cdots = 0$, by (4.421) and (4.431); $A_{p^2} = p^{-7}$, by (4.432); and

$$A_p = (p-1)p^{-4},$$

by (4.415). Thus

$$\chi_p = 1 + (p-1)p^{-4} - p^{-7} = (1+p^{-3})(1-p^{-4}). \tag{5.13}$$

We have therefore

$$S = \chi_2 \prod_{p \nmid n} (1-p^{-4}) \prod_{p|n} \{(1+p^{-3})(1-p^{-4})\}$$
$$= \chi_2 \prod (1-p^{-4}) \prod_{p|n} (1+p^{-3}) = \frac{96}{\pi^4}\chi_2 \prod_{p|n}(1+p^{-3}),$$

since

$$\prod (1-p^{-4}) = \frac{16}{15\zeta(4)} = \frac{96}{\pi^4}.$$

If n is odd,

$$\rho_8(n) = \frac{\pi^4 n^3}{\Gamma(4)} \frac{96}{\pi^4} \prod_{p|n}(1+p^{-3}) = 16\sigma_3(n), \tag{5.14}$$

where $\sigma_3(n)$ is the sum of the cubes of the divisors of n. If n is even,

$$\rho_8(n) = 16n^3(1-2^{-3}) \prod_{p|n}(1+p^{-3}) = 16\{\sigma_3'(n) - \sigma_3''(n)\}, \tag{5.15}$$

where $\sigma_3'(n)$ and $\sigma_3''(n)$ are the sums of the cubes of the even and odd divisors respectively. These are Jacobi's well-known results, proved at present, however, only when n is not divisible by any square.

5.2. Proceeding to the general case, suppose that

$$n = 2^\alpha \varpi^a \varpi'^{a'} \cdots \qquad (\alpha \geqq 0;\ a, a', \cdots > 0) \tag{5.21}$$

and consider first A_{2^λ}.

If $\lambda = 1$, $A_{2^\lambda} = 0$, by 4.31. If λ is odd and greater than 1, $A_{2^\lambda} = 0$, by (4.321), unless $\nu = n/2^{\lambda-3} \equiv 0 \pmod 4$, i.e. unless $n \equiv 0 \pmod{2^{\lambda-1}}$, or unless $\lambda \leqq \alpha + 1$. If this condition is satisfied, and $n = 2^\alpha N$, so that N is odd, we have

$$A_{2^\lambda} = 2^{-3(\lambda-1)} e^{-2^{\alpha+1-\lambda} N\pi i},$$

by (4.3223); and so

$$A_{2^\lambda} = 2^{-3(\lambda-1)} \quad (\lambda < \alpha+1), \qquad A_{2^\lambda} = -2^{-3(\lambda-1)} \quad (\lambda = \alpha+1). \tag{5.22}$$

On the other hand, if λ is even, $A_{2^\lambda} = 0$, by (4.331), unless $n \equiv 0 \pmod{2^{\lambda-2}}$,

[16] Following Landau, I write $p \mid n$ for 'p is a divisor of n' and $p \nmid n$ for 'p is not a divisor of n'.

i.e. unless $\lambda \leqq \alpha + 2$. If this condition is satisfied we have, by (4.3322),

$$A_{2^\lambda} = 2^{-3(\lambda-1)} \cos(2^{\alpha+1-\lambda} N\pi).$$

The cosine is 1 if $\lambda < \alpha + 1$, -1 if $\lambda = \alpha + 1$, and 0 if $\lambda = \alpha + 2$. Thus the equations (5.22) still hold for even values of λ. We have therefore

$$\chi_2 = 1 + 0 + 2^{-3} + 2^{-6} + \cdots + 2^{-3(\alpha-1)} - 2^{-3\alpha}, \tag{5.23}$$

the zero term corresponding to $\lambda = 1$.

Next suppose that p is odd. If p is not an ω, $\chi_p = 1 - p^{-4}$, as before. If $p = \omega$ and $\lambda < a + 1$, $n \equiv 0 \pmod{\omega^\lambda}$, and

$$A_{\omega^\lambda} = (\omega - 1)\omega^{-3\lambda-1},$$

by (4.415), (4.4231), or (4.433). If $\lambda = a + 1$,

$$A_{\omega^\lambda} = -\omega^{-3\lambda-1} = -\omega^{-3a-4},$$

by (4.422), (4.411), and (4.432). And if $\lambda > a + 1$,

$$A_{\omega^\lambda} = 0,$$

by (4.421) and (4.431). Thus

$$\begin{aligned} \chi_\omega &= 1 + (\omega-1)\omega^{-4} + (\omega-1)\omega^{-7} + \cdots + (\omega-1)\omega^{-3a-1} - \omega^{-3a-4} \\ &= (1 - \omega^{-4})(1 + \omega^{-3} + \omega^{-6} + \cdots + \omega^{-3a}). \end{aligned} \tag{5.24}$$

From (5.23) and (5.24) it follows, as at the end of 5.1, that

$$\rho_8(n) = 16n^3\{1 + 2^{-3} + \cdots + 2^{-3(\alpha-1)} - 2^{-3\alpha}\} \prod_\omega (1 + \omega^{-3} + \cdots + \omega^{-3a}),$$

it being understood that the factor in curly brackets is to be replaced by unity when $\alpha = 0$; and it is easily verified that the formulas (5.14) and (5.15) are still correct.

6. Summation of the singular series when $s = 5$ and n has no squared factors

6.1. Suppose next that $s = 5$ and that n is not divisible by any square, and first that $p = 2$. Then $A_{16} = A_{32} = \cdots = 0$, by (4.321) and (4.331). And $A_8 = 0$, by (4.321), unless $n \equiv 1 \pmod 4$, in which case, by (4.3223),

$$A_8 = 2^{-3} e^{-\frac{1}{4}(n-5)\pi i}.$$

Thus $A_8 = -\frac{1}{8}$ if $n \equiv 1 \pmod 8$, $A_8 = \frac{1}{8}$ if $n \equiv 5 \pmod 8$, and otherwise $A_8 = 0$.

Next,

$$A_4 = 2^{-3/2} \cos(\tfrac{1}{2} n\pi - \tfrac{5}{4}\pi),$$

by (4.3322), so that $A_4 = -\frac{1}{4}$ if $n \equiv 1 \pmod 4$ and $A_4 = \frac{1}{4}$ otherwise. Finally $A_2 = 0$ in all cases, by 4.31.

Collecting these results we find that

(6.11) $\chi_2 = \frac{5}{8}(n \equiv 1), \qquad \chi_2 = \frac{5}{4}(n \equiv 2, 3, 6, 7), \qquad \chi_2 = \frac{7}{8}(n \equiv 5),$

the congruences being to modulus 8.

If p is odd and $p \nmid n$, we have

$$A_p = \left(\frac{n}{p}\right)\frac{1}{p^2}, \qquad A_{p^2} = A_{p^3} = \cdots = 0,$$

by (4.412), (4.421), and (4.431). If p is odd and $p|n$, we have

$$A_p = 0, \qquad A_{p^2} = -p^{-4}, \qquad A_{p^3} = A_{p^4} = \cdots = 0,$$

by (4.416), (4.432), (4.421), and (4.431). Thus

$$\chi_p = 1 + \left(\frac{n}{p}\right)\frac{1}{p^2} \quad (p \nmid n), \qquad \chi_p = 1 - \frac{1}{p^4} \quad (p|n). \tag{6.12}$$

If $n \not\equiv 1 \pmod 4$, we have

$$\rho_5(n) = \frac{\pi^{5/2} n^{3/2}}{\Gamma(5/2)} \frac{5}{4} \prod_{p \nmid n} \left\{1 + \left(\frac{n}{p}\right)\frac{1}{p^2}\right\} \prod_{p|n} \left(1 - \frac{1}{p^4}\right)$$
$$= \frac{5}{3}\pi^2 n^{3/2} \prod \left(1 - \frac{1}{p^4}\right) \prod_{p \nmid n} \left\{1 - \left(\frac{n}{p}\right)\frac{1}{p^2}\right\}^{-1}.$$

Also

$$\prod \left(1 - \frac{1}{p^4}\right) = \frac{16}{15\zeta(4)} = \frac{96}{\pi^4}, \qquad \prod_{p \nmid n} \left\{1 - \left(\frac{n}{p}\right)\frac{1}{p^2}\right\}^{-1} = \sum \left(\frac{n}{m}\right)\frac{1}{m^2},$$

where m runs through all odd values prime to n. Hence finally

$$\rho_5(n) = \frac{160}{\pi^2} n^{3/2} \sum \left(\frac{n}{m}\right)\frac{1}{m^2}. \tag{6.13}$$

If $n \equiv 1 \pmod 8$ the value of χ_2 is $\frac{5}{8}$ instead of $\frac{5}{4}$; and if $n \equiv 5 \pmod 8$ it is $\frac{7}{8}$. In these cases the numerical factor 160 must be replaced by 80 and by 112 respectively.

These are the results of Eisenstein, subsequently proved by Smith and Minkowski by means of the arithmetical theory of quadratic forms. The series (6.13) is easily summed in a finite form, by methods due to Dirichlet and to Cauchy. I have nothing to add to this part of the discussion.

7. The general case when $s = 5$

7.1. So far it has not been necessary to distinguish between one type of representation and another. At this stage the distinction between "primitive" and "imprimitive" representations becomes of importance.

A representation

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$$

is said to be *imprimitive* if x_1, x_2, x_3, x_4, x_5 possess a common factor, and *primitive* in the contrary case. It is plain that imprimitive representations can exist only when n is divisible by a square. When $s = 8$ (and the remark applies equally when s is 2, 4, or 6) the distinction is, for our purposes, irrelevant, even when n is divisible by a square: the formulas (5.14) and (5.15) are valid in any case. But when $s = 5$ the distinction is important. It will be remembered in fact, by anyone familiar with the work of Minkowski and Smith, that the right-hand side of (6.13) represents, in general, not the total number of representations but the number of primitive representations. Our series (2.26), on the other hand, gives the total number of representations; and its relation to the Smith-Minkowski series must therefore generally be more intricate than in the simplest case treated in 6.1.

The theorem which I shall prove is as follows:

The sum of the series

$$\frac{C}{\pi^2} n^{3/2} \sum \left(\frac{n}{m}\right) \frac{1}{m^2}, \tag{7.11}$$

where m runs through all odd numbers prime to n, and

$$C = 80\,(n \equiv 0, 1, 4), \quad C = 160\,(n \equiv 2, 3, 6, 7), \quad C = 112\,(n \equiv 5),$$

the congruences being to modulus 8, is $\bar{r}_5(n)$, the number of primitive representations of n.

We shall require the following

Lemma. *If*[17]

$$r(n) = \sum \phi\left(\frac{n}{q^2}\right), \tag{7.12}$$

where q^2 runs through all squared divisors of n, then

$$\phi(n) = \bar{r}(n). \tag{7.13}$$

To prove this, suppose first that n is divisible by p^2, but by no other square. Then

$$\phi(n) = r(n) - \phi\left(\frac{n}{p^2}\right) = r(n) - r\left(\frac{n}{p^2}\right) = \bar{r}(n).$$

Next, if n is divisible by p^2, p'^2, and $(pp')^2$, where $p' \neq p$, but by no other square, we have

$$\phi(n) = r(n) - \phi\left(\frac{n}{p^2}\right) - \phi\left(\frac{n}{p'^2}\right) - \phi\left(\frac{n}{p^2 p'^2}\right)$$

$$= r(n) - \bar{r}\left(\frac{n}{p^2}\right) - \bar{r}\left(\frac{n}{p'^2}\right) - r\left(\frac{n}{p^2 p'^2}\right) = \bar{r}(n).$$

[17] In what follows I omit the suffix 5 in $r_5(n)$, etc.

A similar proof applies if $p = p'$; and it is plain that a repetition of the argument leads to a general proof of the lemma.

7.2. Suppose first that n is congruent to 2, 3, 6, or 7, so that n is not divisible by 4 and $\chi_2 = \frac{5}{4}$, by (6.11). If we write

$$n = 2^\alpha N = 2^\alpha \omega^a \omega'^{\,a'} \cdots \qquad (\alpha = 0, 1), \tag{7.21}$$

the value of χ_p requires reconsideration when p is an ω and $a > 1$. Using the formulas of 4.4, we obtain the following results.

If $a = 2b + 1$ then

$$A_\omega = 0, \quad A_{\omega^2} = (\omega - 1)\omega^{-4}, \quad A_{\omega^3} = 0, \quad A_{\omega^4} = (\omega - 1)\omega^{-7},$$
$$\cdots, \quad A_{\omega^{2b-1}} = 0, \quad A_{\omega^{2b}} = (\omega - 1)\omega^{-3b-1},$$
$$A_{\omega^{2b+1}} = 0, \quad A_{\omega^{2b+2}} = -\omega^{-3b-4}, \quad A_{\omega^{2b+3}} = A_{\omega^{2b+4}} = \cdots = 0.$$

If $a = 2b$, the values of the A's, up to $A_{\omega^{2b}}$, are as above, but

$$A_{\omega^{2b+1}} = \left(\frac{\nu}{\omega}\right)\omega^{-3b-2}, \quad A_{\omega^{2b+2}} = A_{\omega^{2b+3}} = \cdots = 0,$$

where $\nu = \omega^{-a} n$. We thus find that

$$\chi_\omega = \frac{(1 - \omega^{-4})(1 - \omega^{-3b-3})}{1 - \omega^{-3}} \tag{7.22}$$

if a is odd, and

$$\chi_\omega = \frac{(1 - \omega^{-4})(1 - \omega^{-3b})}{1 - \omega^{-3}} + \left\{1 + \left(\frac{\nu}{\omega}\right)\frac{1}{\omega^2}\right\}\omega^{-3b} \tag{7.23}$$

if a is even.

Suppose now that $n = 2^\alpha \omega^a \omega'^{\,a'} \cdots = \omega^2 d$, where d has no squared factor. Then the odd primes p fall into four classes characterized as follows.

(i) $p = \boldsymbol{p}$, $\boldsymbol{p} \nmid n$. In this case

$$\chi_{\boldsymbol{p}} = 1 + \left(\frac{n}{\boldsymbol{p}}\right)\boldsymbol{p}^{-2}. \tag{7.241}$$

(ii) $p = \omega_1$, $\omega_1 \nmid \omega$, $\omega_1 | d$. In this case

$$\chi_{\omega_1} = 1 - \omega_1^{-4}. \tag{7.242}$$

(iii) $p = \omega_2$, $\omega_2 | \omega$, $\omega_2 \nmid d$. In this case a is even, say equal to $2b$, and

$$\chi_{\omega_2} = (1 - \omega_2^{-4})\left\{1 + \omega_2^{-3} + \cdots + \omega_2^{-3b+3} + \frac{\omega_2^{-3b}}{1 - \left(\frac{\nu}{\omega_2}\right)\frac{1}{\omega_2^2}}\right\}, \tag{7.243}$$

by (7.23).

(iv) $p = \omega_3$, $\omega_3 | \omega$, $\omega_3 | d$. In this case a is odd, say equal to $2b + 1$, and

$$\chi_{\omega_3} = (1 - \omega_3^{-4})(1 + \omega_3^{-3} + \cdots + \omega_3^{-3b}), \tag{7.244}$$

by (7.22). And we have

(7.25) $$S = \tfrac{5}{4}\prod \chi_p = \tfrac{5}{4}\prod \chi_p \prod \chi_{\omega_1} \prod \chi_{\omega_2} \prod \chi_{\omega_3}.$$

7.3. We now multiply out the product (7.25), treating the second factors of χ_{ω_2} and χ_{ω_3} each as a sum of $b+1$ separate terms. We thus obtain

(7.31) $$S = \tfrac{5}{4}\prod_p (1 - p^{-4}) \prod_p \gamma_p(n) \sum_{\lambda, \mu} \left(\prod_{\omega_2} \theta\omega_2^{-3\lambda} \prod_{\omega_3} \omega_3^{-3\mu}\right),$$

where

(7.311) $$\gamma_p(n) = \left\{1 - \left(\frac{n}{p}\right)\frac{1}{p^2}\right\}^{-1};$$

$\lambda \leqq b_2$, if $\omega_2^{a_2} = \omega_2^{2b_2}$ is the highest power of ω_2 which divides n; $\mu \leqq b_3$, if $\omega_3^{a_3} = \omega_3^{2b_3+1}$ is the highest power of ω_3 which divides n; and θ is an additional factor which is equal to 1 unless $\lambda = b_2$, and then to

$$\gamma_{\omega_2}(\nu) = \gamma_{\omega_2}(\omega_2^{-a_2} n).$$

If we denote the product which appears under the sign of summation in (7.31) by $\sigma_{\lambda, \mu}$, we have

(7.32) $$\rho(n) = \frac{4\pi^2}{3} n^{3/2} S = \frac{160}{\pi^2} n^{3/2} \prod_p \gamma_p(n) \sum \sigma_{\lambda, \mu} = \sum \rho_{\lambda, \mu},$$

say.

Suppose first that λ does not, for any ω_2, assume its maximum value b_2, so that all the θ's in $\sigma_{\lambda, \mu}$ are equal to unity; and write

(7.34) $$\psi(n) = \frac{160}{\pi^2} n^{3/2} \sum \left(\frac{n}{m}\right) \frac{1}{m^2},$$

so that

$$\psi(n) = \frac{160}{\pi^2} n^{3/2} \prod_p \gamma_p(n),$$

the product extending over all odd primes which do not divide n. Then

$$\rho_{\lambda, \mu} = \frac{160}{\pi^2} n^{3/2} \left(\prod_{\omega_2} \omega_2^{-2\lambda} \prod_{\omega_3} \omega_3^{-2\mu}\right)^{3/2} \prod_p \gamma_p(n) = \psi\left(\frac{n}{q^2}\right),$$

where

$$q^2 = \prod_{\omega_2} \omega_2^{-2\lambda} \prod_{\omega_3} \omega_3^{-2\mu}$$

is a typical square divisor of n, division by which does not eliminate completely any prime factor of n.

This transformation would not, as it stands, be valid if $\lambda = b_2$ for some ω_2, since there are then certain primes ω_2 which divide n and not n/q^2. But with each of these primes ω_2 there is associated an additional factor $\theta = \gamma_{\omega_2}(\nu)$ in $\sigma_{\lambda, \mu}$, and these factors provide exactly the corrective required. We have therefore in any case $\rho_{\lambda,\mu} = \psi(n/q^2)$ and

$$r(n) = \rho(n) = \sum \psi\left(\frac{n}{q^2}\right),$$

the summation extending over all square divisors of n. And therefore, by (7.13), $\psi(n) = \bar{r}(n)$, the result required.

Our theorem is thus proved when n is congruent to 2, 3, 6, or 7 to modulus 8. In order to prove it when n is congruent to 1 or 5, we have only to write $\frac{5}{8}$ or $\frac{7}{8}$ instead of $\frac{5}{4}$ throughout our argument. It is only when n is divisible by 4 that further discussion is required.

7. 4. We have now

$$n = 2^{\alpha} N = 2^{\alpha} \omega^{a} \omega'^{a'} \cdots \qquad (\alpha \geqq 2).$$

The value of χ_p, when p is odd, is the same as before. The value of χ_2 may be calculated by means of the results of 4.3; and we find that

$$\chi_2 = 1 - \frac{1}{4} - \frac{1}{4.8} - \frac{1}{4.8^2} - \cdots - \frac{1}{4.8^{\beta-1}} + \frac{1}{4.8^{\beta}}, \tag{7.41}$$

where α is odd and equal to $2\beta + 1$ or even and equal to 2β.

Let us denote by $r^*(n)$ the number of representations of n which are *primitive so far as* 2 *is concerned*, that is to say in which x_1, x_2, x_3, x_4, and x_5 are not all even. It is plain that

$$r(n) = r^*(n) + r^*\left(\frac{n}{4}\right) + \cdots + r^*\left(\frac{n}{4^{\beta}}\right), \tag{7.42}$$

and that

$$r^*(n) = \sum \bar{r}\left(\frac{n}{q^2}\right), \tag{7.43}$$

where now the summation applies to all *odd* square divisors of n. Further, as in 7.1, we can show that *if*

$$r^*(n) = \sum \phi\left(\frac{n}{q^2}\right), \tag{7.44}$$

where the summation applies to all odd square divisors of n, *then*

$$\phi(n) = \bar{r}(n). \tag{7.45}$$

Bearing these remarks in mind, we can complete the proof of the theorem as follows. Since $\rho(n)$, $\rho(\frac{1}{4}n)$, $\cdots$ differ only in the factor χ_2 and the outside power of n, $\frac{1}{4}n$, $\cdots$, we have, by (7.41),

$$\begin{aligned}\rho(n) - \rho\left(\frac{n}{4}\right) = \tfrac{4}{5}\rho\left(\frac{n}{4^{\beta}}\right)\Bigg\{8^{\beta}\left(1 - \frac{1}{4} - \cdots - \frac{1}{4.8^{\beta-1}} + \frac{1}{4.8^{\beta}}\right) \\ - 8^{\beta-1}\left(1 - \frac{1}{4} - \cdots - \frac{1}{4.8^{\beta-2}} + \frac{1}{4.8^{\beta-1}}\right)\Bigg\} \\ = 4.8^{\beta-1}\rho\left(\frac{n}{4^{\beta}}\right).\end{aligned}$$

But, by (7.42), we have

$$r^*(n) = r(n) - r\left(\frac{n}{4}\right) = \rho(n) - \rho\left(\frac{n}{4}\right) = 4.8^{\beta-1}\rho\left(\frac{n}{4^\beta}\right) = 4.8^{\beta-1}r\left(\frac{n}{4^\beta}\right);$$

and therefore, by our previous results,

$$r^*(n) = 4.8^{\beta-1}\sum \bar{r}\left(\frac{n}{4^\beta q^2}\right) = 4.8^{\beta-1}\sum \psi\left(\frac{n}{4^\beta q^2}\right) = \tfrac{1}{2}\sum \psi\left(\frac{n}{q^2}\right),$$

the summation applying to all square divisors of $n/4^\beta$, or, what is the same thing, to all odd square divisors of n. Hence, by (7.45),

$$\psi(n) = 2\bar{r}(n).$$

This completes the proof of the theorem. It is easily verified that the results are in complete agreement with those of Smith.[18]

8. Concluding remarks

8.1. I have assumed, throughout this paper, that $s \leqq 8$; and it is well known that the analogous results when $s > 8$ are false.

The analysis of the paper breaks down, when $s > 8$, in one section only, namely Section 3. We can still form the singular series, and sum it by methods differing only in detail from those of Sections 4–7. We obtain a simple function of the divisors of n when s is even, a series of the Smith-Minkowski type when s is odd; and this series can still be summed in terms of the quadratic residues and non-residues of n. We can still prove, moreover, that the sum of the singular series behaves, in respect to the fundamental transformations of the modular subgroup Γ_3, exactly like the appropriate power of the theta-function ϑ, and that the function corresponding to $\eta(\tau)$ is an invariant of the group. What we cannot prove is that $\eta(\tau)$ is bounded; and the conclusion which would follow from this, namely that $\eta(\tau)$ is constant, is in fact false.

We have still, however, all the materials for a complete solution of the problem. But it is necessary to replace the analysis of Section 3 by a more complex discussion in which we deal not with a single invariant but with a linear combination of invariants, among which that represented by the sum of the singular series is the first and most important. And our conclusion will be that the number of representations of n is the sum of a function of the types considered in this paper and of a number of other arithmetical functions defined in a more recondite manner. Some of these functions have already appeared in the work of Liouville, Glaisher, and Mordell. If I do not pursue

[18] See in particular pp. 673 et seq. of the second volume of his *Collected Papers*.

this subject further, it is because such developments seem to be a part of Mr. Mordell's researches rather than of mine.

There is another question which arises more naturally out of my own researches. The singular series or principal invariant yields in any case an *asymptotic* formula for $r_s(n)$, valid without restriction on s. But, with the entry of asymptotic formulas, the peculiar interest of squares as such departs, and the problem becomes merely a somewhat trivial case of the much larger problem usually described as Waring's problem, and so of the investigations which Mr. Littlewood and I are publishing elsewhere.